

Polynomial parametrizations of length 4 Büchi sequences

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Abstract

Büchi's problem asks whether there exists a positive integer M such that any sequence (x_n) of at least M integers, whose second difference of squares is the constant sequence (2), satisfies $x_n^2 = (x + n)^2$ for some $x \in \mathbb{Z}$. A positive answer to Büchi's problem would imply that there is no algorithm to decide whether or not an arbitrary system of quadratic diagonal forms over \mathbb{Z} can represent an arbitrary given vector of integers. We give explicitly an infinite family of polynomial parametrizations of non-trivial length 4 Büchi sequences of integers. In turn, these parametrizations give an explicit infinite family of curves (which we suspect to be hyperelliptic) with the following property: any integral point on one of these curves would give a length 5 non-trivial Büchi sequence of integers (it is not known whether any such sequence exists).

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1 Introduction

Since the projective surface with affine equations

$$x_4^2 - 2x_3^2 + x_2^2 = x_3^2 - 2x_2^2 + x_1^2 = 2 \quad (1)$$

is a Segre surface (a Del Pezzo surface of degree 4), its \mathbb{Q} -rational points can be parametrized. Let X_4 be the (affine) variety defined by Equations (1) (*Büchi equations*). Having Büchi's problem in mind, we would like to characterize the set of integer points on X_4 (actually a cofinite set of the set of integer points would be enough). There exists extensive literature about rational surfaces, but there seem to be few results about polynomial parametrizations over \mathbb{Z} .

Let A be a commutative ring with unit. A sequence (x_1, \dots, x_ℓ) of elements of A is called a *Büchi sequence over A* if its second difference is the constant sequence (2): for each $i \in \{1, \dots, \ell - 2\}$ it satisfies

$$x_i^2 - 2x_{i+1}^2 + x_{i+2}^2 = 2.$$

If A has characteristic 0, we will call *trivial Büchi sequence* any sequence satisfying: there exists $x \in A$ such that $x_i^2 = (x+i)^2$ for all $i = 1, \dots, \ell$. Büchi's problem over A asks whether there exists an integer M such that no non-trivial Büchi sequence of length at least M exists. If such an M exists, we call $M(A)$ the smallest one, and $M_f(A)$ the least M such that there are only finitely many non-trivial Büchi sequences of length M . Hence, if one proves that $M_f(A)$ exists, then one obtains automatically a positive answer to Büchi's problem for some $M \geq M_f(A)$.

If A is a ring of functions, then it is required that at least one x_i is transcendental over the prime subring of A (also in the case where A has positive characteristic, the concept of trivial sequence has to be adapted - see for example [PPV]).

Büchi got interested in this problem when he realized that from a positive answer to it he would be able to prove that there is no algorithm to decide whether or not an arbitrary system of quadratic diagonal forms can represent an arbitrary given vector of integers (which, if true, would be one of the strongest forms of the negative answer to Hilbert's tenth problem).

Büchi's problem remains open for the integers, but P. Vojta [Vo] showed that $M_f(\mathbb{Q})$ would be 8 (actually the proof goes through for any number field) if Bombieri conjecture would be true for surfaces. It is striking that even though we cannot prove that Büchi's problem has a positive answer, no non-trivial Büchi sequence of length even just 5 over \mathbb{Z} is known to exist. Actually Büchi had conjectured that $M(\mathbb{Z}) = 5$.

Büchi's problem is known to have a positive answer over many usual rings of functions: fields of complex and p -adic complex meromorphic functions ([Vo], [P3]), function fields in characteristic zero or large enough ([Vo],[PV2],[PV3],[ShV]). Most of these results have been generalized to stronger forms in [P1] and [P3]. A higher power version of Büchi's problem was introduced in [PV1] (looking at k -th difference of k -th powers), but very few results have been obtained so far: it has a positive answer for any power over finite fields \mathbb{F}_p - see [P2] - and for cubes over rings of polynomials - see [PV4]. We refer to the surveys [PPV] for results in these directions.

In [A], [Bre] and [BB] one can find results about analogues of Büchi's problem where the constant sequence (2) is replaced by another constant sequence.

Büchi sequences of length 3 are not difficult to characterize over \mathbb{Q} , and with some divisibility conditions one obtains a complete characterization of sequences over \mathbb{Z} - they are infinitely many - see [H2, Theorem 2.1] or [PPV, Section 7]. We also know a characterization over \mathbb{Z} that does not require any divisibility condition (i.e. without any reference to \mathbb{Q}) - see [SaV].

Already Hensley [H2] knew that there exist infinitely many length 4 Büchi sequences of integers. Obtaining a "good" characterization for (a cofinite subset of the set of all) length 4 sequences of integers could be a key step for solving Büchi's problem: proving that no sequences of length 4 (but finitely many) can be extended to length 5 could then be quite easier, and would prove that $M_f(\mathbb{Z}) = 5$.

This work presents an effort to characterize all but finitely many Büchi sequences of length 4 over the integers. The idea comes from an unpublished paper by D. Hensley [H2] from the early eighties, where a polynomial parametrization of degree 3 for length 4 integer sequences is described, and from a paper by R. G. E. Pinch [Pi] from 1993 where he lists many length 4 non-trivial Büchi sequences and shows that none of them can be extended to length 5 sequences.

In Section 2 we give explicitly a birational map ζ on X_4 , of infinite order, with the following property (proved in Section 3):

Theorem 1.1. *For any $t \in \mathbb{Z}$ and any non-negative integer n , the n -th iterate*

$$\zeta^{(n)}(t, t+1, t+2, t+3)$$

has the form

$$\xi(n, t) = (\xi_1(n, t), \xi_2(n, t), \xi_3(n, t), \xi_4(n, t))$$

where each $\xi_i(n, t)$ is a polynomial in t of degree $2n+1$.

The polynomials $\xi_i(n, t) \in \mathbb{Z}[t]$ are given explicitly in Section 3: they satisfy the following linear second order recurrence relation: for all $n \geq 0$

$$\xi(n+2, t) = f(t)\xi(n+1, t) - \xi(n, t),$$

where $f(t) = 2t^2 + 10t + 10$, and with initial values

$$\begin{aligned}\xi(0, t) &= (t+1, t+2, t+3, t+4) \\ \xi(1, t) &= (2t^3 + 12t^2 + 19t + 6, 2t^3 + 14t^2 + 31t + 23, 2t^3 + 16t^2 + 41t + 32, 2t^3 + 18t^2 + 49t + 39).\end{aligned}$$

In other words, there are infinitely many non-trivial parametrizations of Büchi sequences of length 4 over the integers (one can prove that they are *essentially* distinct, as they give rise to distinct sequences). Iterating ζ on known rational parametrizations over \mathbb{Q} gives rise to infinitely many rational parametrizations (points of $X_4(\mathbb{Q}(t))$).

The parametrization $\xi(1, t)$ is actually the one that D. Hensley [H2] had found in the early eighties, and no other polynomial parametrization seemed to be known to this day.

In Section 4, we present two more polynomial parametrizations over \mathbb{Z} , of degree 4, and one polynomial parametrization over \mathbb{Q} , also of degree 4. Then we give a non-exhaustive list of rational parametrizations which are not polynomial parametrizations. Computationally, it seems that there are no other polynomial parametrizations than the ones we already found, but we cannot prove it.

Also we cannot prove the following, which seems to be true computationally: none of these parametrizations can represent an integer solution that extends to a length 5 Büchi sequence. Indeed, consider for example the sequence $\xi(1, t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ given above. Asking for this sequence to extend, for some fixed integer t , to a length 5 sequence is asking whether either $2x_4^2 - x_3^2 + 2$ (extension to the right) or $2x_1^2 - x_2^2 + 2$ (extension to the left) is a square. Namely: do one of the following curves

$$y^2 = 4t^6 + 80t^5 + 620t^4 + 2400t^3 + 4905t^2 + 5020t + 2020$$

or

$$y^2 = 4t^6 + 40t^5 + 120t^4 - 595t^2 - 970t - 455$$

have an integer point?

In Section 7, we list all integer solutions that we found and that we were not able to *parametrize* (i.e.: they seem not to belong to the image of a polynomial parametrization). With first term at most 1052749, they are 121 (counting only the strictly increasing sequences of positive integers) and we do not know whether or not we are missing finitely many. From the figure at the end of the section, it *seems* clear that the quantity of points that we are “missing” is decreasing exponentially with respect to the size of the points. None of these (non-parametrized) points can extend to a length 5 solution, as is easily verified with a computer software.

The symbol \dagger in the text will mean that we are using a computer software for the formal computation. We have used exclusively the open source software Xcas 0.8.6 and 0.9.0 for all our computations: Giac/Xcas, Bernard Parisse et Rene De Graeve, version 0.8.6 (2010)
<http://www-fourier.ujf-grenoble.fr/~parisse/giac.fr.html>

2 Some birational maps on X_4

Notation 2.1. 1. Denote by $\text{Bir}(X_4)$ the group of birational maps on X_4 .

2. Let τ and μ_i , $i = 1, 2, 3, 4$, denote the following automorphisms of X_4 :

$$\mu_1(a, b, c, d) = (-a, b, c, d) \quad \mu_2(a, b, c, d) = (a, -b, c, d)$$

$$\mu_3(a, b, c, d) = (a, b, -c, d) \quad \mu_4(a, b, c, d) = (a, b, c, -d)$$

and

$$\tau(a, b, c, d) = (d, c, b, a).$$

Observe that each μ_i is an odd function.

3. We will call trivial involution on X_4 any map from the subgroup Γ_1 of $\text{Bir}(X_4)$ generated by the set $\{\mu_1, \mu_2, \mu_3, \mu_4, \tau\}$.
4. Write Γ_0 for the group generated by $\{\mu_1, \mu_2, \mu_3, \mu_4\}$.
5. Write $\mu_{ij} = \mu_i \mu_j$ and $\mu_{ijk} = \mu_i \mu_j \mu_k$ for any $i, j, k \in \{1, 2, 3, 4\}$.

Remark 2.2. 1. For all $i \neq j$ we have $\mu_i \mu_j = \mu_j \mu_i$, hence Γ_0 is isomorphic to $(\mathbb{Z}_2)^4$.

2. We have $\tau \mu_1 = \mu_4 \tau$ and $\tau \mu_2 = \mu_3 \tau$.
3. We have $\tau \mu_{14} = \mu_{14} \tau$ and $\tau \mu_{23} = \mu_{23} \tau$.
4. For each i , $\tau \mu_i$ has order 4.

Lemma 2.3. For all i , we have $\tau \mu_i \tau = \mu_{\sigma(i)}$, where σ stands for the permutation $(1\ 4)(2\ 3) \in S_4$. Hence the group H is normal in Γ_1 and the group Γ_1 is a semi-direct product $\Gamma_0 \rtimes \langle \tau \rangle$.

Proof. This is clear from the above remarks. □

Next we define a rational map φ on X_4 that will turn out to be an involution.

Notation 2.4. 1. We will consider the map $\varphi: F^4 \rightarrow F^4$ defined by

$$(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \left(\frac{p_1}{q}, \frac{p_2}{q}, \frac{p_3}{q}, \frac{p_4}{q} \right)$$

where

$$q(a, b, c, d) = (b - c)^2(a - 2b + c)$$

$$\begin{aligned} p_1(a, b, c, d) = & -2ab^3 + ab^2c + 2ab^2d + 4abc^2 - 5abcd + ab - 2ac^3 + 2ac^2d + ac - ad + 3b^4 \\ & - 2b^3c - 3b^3d - 6b^2c^2 + 8b^2cd + b^2 + 4bc^3 - 4bc^2d - 5bc + bd + c^2 + cd - 2 \end{aligned}$$

$$\begin{aligned} p_2(a, b, c, d) = & -2ab^2c + 5abc^2 - 2abcd + 2ab - 2ac^3 + ac^2d \\ & - ad + 3b^3c - 8b^2c^2 + 3b^2cd - 2b^2 + 4bc^3 - 2bc^2d - bc + 2bd - 2 \end{aligned}$$

$$\begin{aligned} p_3(a, b, c, d) = & -2ab^3 + 5ab^2c - 2ab^2d - 2abc^2 + abcd \\ & + 3ab - ac - ad + 3b^4 - 8b^3c + 3b^3d + 4b^2c^2 - 2b^2cd - 3b^2 - bc + 3bd + c^2 - cd - 2 \end{aligned}$$

$$\begin{aligned} p_4(a, b, c, d) = & -3ab^3 + 8ab^2c - 3ab^2d - 4abc^2 + 2abcd + 4ab - 2ac - ad + 4b^4 - 10b^3c + 4b^3d \\ & + 2b^2c^2 - 2b^2cd - 4b^2 + 5bc^3 - 2bc^2d - bc + 4bd - 2c^4 + c^3d + 2c^2 - 2cd - 2. \end{aligned}$$

Observe that φ is an odd function (as q is an odd function and p_i are even functions).

2. Write $\zeta = \varphi \tau \mu_{14}$.

Lemma 2.5. *The map φ is a rational map on X_4 .*

Proof. One needs to replace formally (\dagger) a^2 by $2b^2 - c^2 + 2$ and d^2 by $2c^2 - b^2 + 2$ in the expressions $(\varphi_1^2 - 2\varphi_2^2 + \varphi_3^2)(a, b, c, d)$ and $(\varphi_2^2 - 2\varphi_3^2 + \varphi_4^2)(a, b, c, d)$. \square

Lemma 2.6. *The map φ is an involution.*

Proof. For each i , after substituting formally (\dagger) x_4^2 by $2x_3^2 - x_2^2 + 2$ and x_3^2 by $2x_2^2 - x_1^2 + 2$ in $\varphi_i(\varphi(x_1, x_2, x_3, x_4))$ and doing the obvious simplifications (\dagger) , one obtains x_i . Note that it is not hard to prove this lemma without the help of a computer, by using the fact (verifiable by hand) that

$$(\varphi_1 - 2\varphi_2 + \varphi_3)(a, b, c, d) = a - 2b + c$$

and

$$(\varphi_2 - 2\varphi_3 + \varphi_4)(a, b, c, d) = b - 2c + d.$$

\square

Observe that since φ is birational, also $\zeta = \varphi\tau\mu_{14}$ is birational.

Notation 2.7. *Write Γ for the subgroup of $\text{Bir}(X_4)$ generated by Γ_1 and φ .*

Unfortunately, we do not know the presentation of Γ , but the next two lemmas give us some useful information about it.

Lemma 2.8. *We have $\tau\varphi = \varphi\tau$ and $\tau\zeta = \zeta\tau$.*

Proof. Verifying that $\tau\varphi - \varphi\tau = 0$ needs replacing a^2 by $2b^2 - c^2 + 2$ everywhere it occurs in the expression (\dagger) . Recalling the definition of $\zeta = \varphi\tau\mu_{14}$, we have

$$\tau\zeta\tau = \tau(\varphi\tau\mu_{14})\tau = \varphi\mu_{14}\tau = \varphi\tau\mu_{14} = \zeta.$$

\square

Lemma 2.9. *The map ζ has infinite order.*

Proof. The sequence (u_n) defined for $n \geq 0$ by

$$u_n = \frac{(5 + \sqrt{24})^n - (5 - \sqrt{24})^n}{2\sqrt{24}} \quad (2)$$

is strictly increasing (since $(5 + \sqrt{24})^n$ is strictly increasing and $0 < 5 - \sqrt{24} < 1$). Also it satisfies (\dagger) the recurrence relation

$$u_{n+2} = 10u_{n+1} - u_n, \quad u_0 = 0, \quad u_1 = 1$$

(hence it is a sequence of integers). Therefore, we have

$$0 < u_{n+3} - u_{n+2} = 10(u_{n+2} - u_{n+1}) - (u_{n+1} - u_n) \quad (3)$$

Let us show that for any $n \geq 1$, we have

$$\zeta^{(n)}(1, 2, 3, 4) = u_n(6, 23, 32, 39) - u_{n-1}(1, 2, 3, 4). \quad (4)$$

Since (\dagger)

$$\zeta(1, 2, 3, 4) = (6, 23, 32, 39) \quad \text{and} \quad \zeta^{(2)}(1, 2, 3, 4) = (59, 228, 317, 386),$$

we have

$$\zeta^{(2)}(1, 2, 3, 4) = 10(6, 23, 32, 39) - (1, 2, 3, 4)$$

and Equation (4) is true for $n = 2$. Suppose that it is true up to n . We have

$$\begin{aligned}\zeta^{(n+1)}(1, 2, 3, 4) &= \zeta \circ \zeta^{(n)}(1, 2, 3, 4) \\ &= \zeta(u_n(6, 23, 32, 39) - u_{n-1}(1, 2, 3, 4)) \\ &= u_{n+1}(6, 23, 32, 39) - u_n(1, 2, 3, 4),\end{aligned}$$

where the last equality can be verified with the use of a computer software. This finishes the induction.

Since the second component of $(\zeta^{(n+1)} - \zeta^{(n)})(1, 2, 3, 4)$ is

$$\begin{aligned}(23u_{n+2} - 2u_{n+1}) - (23u_{n+1} - 2u_n) &= 23(u_{n+2} - u_{n+1}) - 2(u_{n+1} - u_n) \\ &= 3(u_{n+2} - u_{n+1}) + 2[10(u_{n+2} - u_{n+1}) - (u_{n+1} - u_n)]\end{aligned}$$

which is strictly positive by Equation (3), this shows that ζ has infinite order. \square

Note that from the above proof we also obtain :

$$\begin{aligned}\zeta^{(n)}(1, 2, 3, 4) &= u_n(6, 23, 32, 39) - u_{n-1}(1, 2, 3, 4) \\ &= (10u_{n-1} - u_{n-2})(6, 23, 32, 39) - (10u_{n-2} - u_{n-1})(1, 2, 3, 4) \\ &= 10(u_{n-1}(6, 23, 32, 39) - u_{n-2}(1, 2, 3, 4)) - (u_{n-2}(6, 23, 32, 39) - u_{n-1}(1, 2, 3, 4)) \\ &= 10\zeta^{(n-1)}(1, 2, 3, 4) - \zeta^{(n-2)}(1, 2, 3, 4)\end{aligned}$$

for all $n \geq 2$.

3 Büchi sequences of length 4 over $\mathbb{Z}[t]$

Notation 3.1. For all $n \geq 0$, write $\xi(n, t) = (\xi_1(n, t), \xi_2(n, t), \xi_3(n, t), \xi_4(n, t))$ where

$$\begin{aligned}\xi_1(n, t) &= \frac{(t^3 + 6t^2 + 9t + 1 + \alpha(t + 1))\beta^n - (t^3 + 6t^2 + 9t + 1 - \alpha(t + 1))\bar{\beta}^n}{2\alpha} \\ \xi_2(n, t) &= \frac{(t^3 + 7t^2 + 16t + 13 + \alpha(t + 2))\beta^n - (t^3 + 7t^2 + 16t + 13 - \alpha(t + 2))\bar{\beta}^n}{2\alpha} \\ \xi_3(n, t) &= \frac{(t^3 + 8t^2 + 21t + 17 + \alpha(t + 3))\beta^n - (t^3 + 8t^2 + 21t + 17 - \alpha(t + 3))\bar{\beta}^n}{2\alpha} \\ \xi_4(n, t) &= \frac{(t^3 + 9t^2 + 24t + 19 + \alpha(t + 4))\beta^n - (t^3 + 9t^2 + 24t + 19 - \alpha(t + 4))\bar{\beta}^n}{2\alpha}\end{aligned}$$

$$\alpha = \sqrt{(t + 1)(t + 2)(t + 3)(t + 4)}$$

and

$$\beta = t^2 + 5t + 5 + \alpha, \quad \bar{\beta} = t^2 + 5t + 5 - \alpha.$$

Theorem 3.2. We have :

1. $\xi(0, t) = (t + 1, t + 2, t + 3, t + 4)$.
2. $\xi(1, t) = (2t^3 + 12t^2 + 19t + 6, 2t^3 + 14t^2 + 31t + 23, 2t^3 + 16t^2 + 41t + 32, 2t^3 + 18t^2 + 49t + 39)$.
3. For all $n \geq 0$

$$\xi(n + 2, t) = f(t)\xi(n + 1, t) - \xi(n, t),$$

where $f(t) = 2t^2 + 10t + 10$.

4. For each $n \geq 0$, $\xi(n, t)$ is a 4-tuple of polynomials of degree $2n + 1$ in the variable t .

5. For all $n \geq 0$ and $t \in \mathbb{Z}$, the sequence $\xi(n, t)$ is a Büchi sequence.

6. For all $n \geq 0$ we have

$$\zeta^{(n)}(t+1, t+2, t+3, t+4) = \xi(n, t).$$

Proof. Items 1, 2, 3 and 5 are easily verified (\dagger). Item 4 then comes immediately from Items 1, 2 and 3 by induction on n . We prove Item 6 by induction on n . For $n = 0$ it is given by Item 1. Suppose it is true up to n . One verifies (\dagger) that

$$\xi(n+1, t) = \zeta(\xi(n, t))$$

hence

$$\xi(n+1, t) = \zeta(\zeta^{(n)}(t+1, t+2, t+3)) = \zeta^{(n+1)}(t+1, t+2, t+3, t+4),$$

which finishes the induction. \square

In the next proposition, we solve the induction in order to find the coefficients of the polynomials $\xi_i(n, t)$.

Proposition 3.3. *If (u_n) is a sequence of integers satisfying $u_{n+2} = \alpha u_{n+1} - u_n$ for each $n \geq 0$, then we have for each $n \geq 1$*

$$u_{2n} = \sum_{k=0}^{n-1} (-1)^{n+k+1} \alpha^{2k} \left(\binom{n+k}{n-k-1} \alpha u_1 - \binom{n+k-1}{n-k-1} u_0 \right)$$

and

$$u_{2n-1} = \alpha^{2n-2} u_2 + \sum_{k=0}^{n-2} (-1)^{n+k+1} \alpha^{2k} \left(\binom{n+k-1}{n-k-1} u_1 + \binom{n+k-1}{n-k-2} \alpha u_0 \right).$$

Proof. The proof is easy and left to the reader. \square

For example, applying the above proposition to $u_n = \xi_1(n, t)$ and $\alpha = f(t) = 2t^2 + 10t + 10$, we obtain

$$\begin{aligned} \xi_1(2n, t) &= \sum_{k=0}^{n-1} (-1)^{n+k+1} 2^{2k} (t^2 + 5t + 5)^{2k} \\ &\quad \left(\binom{n+k}{n-k-1} (4t^5 + 44t^4 + 178t^3 + 322t^2 + 250t + 60) - \binom{n+k-1}{n-k-1} (t+1) \right). \end{aligned}$$

We finish this section with a list of the first three non-trivial parametrizations:

$$\begin{aligned} \xi_1(1, t) &= 2t^3 + 12t^2 + 19t + 6 \\ \xi_2(1, t) &= 2t^3 + 14t^2 + 31t + 23 \\ \xi_3(1, t) &= 2t^3 + 16t^2 + 41t + 32 \\ \xi_4(1, t) &= 2t^3 + 18t^2 + 49t + 39 \end{aligned}$$

$$\begin{aligned} \xi_1(2, t) &= 4t^5 + 44t^4 + 178t^3 + 322t^2 + 249t + 59 \\ \xi_2(2, t) &= 4t^5 + 48t^4 + 222t^3 + 496t^2 + 539t + 228 \\ \xi_3(2, t) &= 4t^5 + 52t^4 + 262t^3 + 634t^2 + 729t + 317 \\ \xi_4(2, t) &= 4t^5 + 56t^4 + 298t^3 + 748t^2 + 879t + 386 \end{aligned}$$

$$\begin{aligned}
\xi_1(3, t) &= 8t^7 + 128t^6 + 836t^5 + 2864t^4 + 5496t^3 + 5816t^2 + 3061t + 584 \\
\xi_2(3, t) &= 8t^7 + 136t^6 + 964t^5 + 3692t^4 + 8256t^3 + 10792t^2 + 7639t + 2257 \\
\xi_3(3, t) &= 8t^7 + 144t^6 + 1084t^5 + 4408t^4 + 10416t^3 + 14248t^2 + 10419t + 3138 \\
\xi_4(3, t) &= 8t^7 + 152t^6 + 1196t^5 + 5036t^4 + 12216t^3 + 17024t^2 + 12601t + 3821
\end{aligned}$$

4 Other parametrizations

Note that by replacing t by t^2 in a polynomial parametrization of degree n , we obtain a polynomial parametrization of degree $2n$. Since there exist non-trivial polynomial parametrizations of any odd degree, there are non-trivial polynomial parametrization of any degree > 2 .

The following

$$\begin{aligned}
P_1(t) &= \frac{t^4 + 17t^3 + 104t^2 + 262t + 204}{4} \\
P_2(t) &= \frac{t^4 + 19t^3 + 138t^2 + 458t + 592}{4} \\
P_3(t) &= \frac{t^4 + 21t^3 + 168t^2 + 602t + 812}{4} \\
P_4(t) &= \frac{t^4 + 23t^3 + 194t^2 + 718t + 984}{4}
\end{aligned}$$

gives a polynomial parametrization $P = (P_1, P_2, P_3, P_4)$ over \mathbb{Q} that takes an integer value for each integer t not congruent to 3 modulo 4. Hence one obtains two *new* polynomial parametrizations over \mathbb{Z} as follows :

$$\begin{aligned}
P_1(2t) &= 4t^4 + 34t^3 + 104t^2 + 131t + 51 \\
P_2(2t) &= 4t^4 + 38t^3 + 138t^2 + 229t + 148 \\
P_3(2t) &= 4t^4 + 42t^3 + 168t^2 + 301t + 203 \\
P_4(2t) &= 4t^4 + 46t^3 + 194t^2 + 359t + 246
\end{aligned}$$

and

$$\begin{aligned}
P_1(4t + 1) &= 64t^4 + 336t^3 + 644t^2 + 525t + 147 \\
P_2(4t + 1) &= 64t^4 + 368t^3 + 804t^2 + 795t + 302 \\
P_3(4t + 1) &= 64t^4 + 400t^3 + 948t^2 + 1005t + 401 \\
P_4(4t + 1) &= 64t^4 + 432t^3 + 1076t^2 + 1179t + 480.
\end{aligned}$$

They are new in the sense that they generate Büchi sequences of integers that were not in the image of any of the $\xi(n, t)$.

The following is another polynomial parametrization over \mathbb{Q} , but it does not reach any integer solution :

$$\begin{aligned}
&\frac{1}{3} (4t^4 + 18t^3 + 14t^2 - 15t - 8) \\
&\frac{1}{3} (4t^4 + 22t^3 + 36t^2 + 19t + 5) \\
&\frac{1}{3} (4t^4 + 26t^3 + 54t^2 + 35t + 2) \\
&\frac{1}{3} (4t^4 + 30t^3 + 68t^2 + 45t + 1) .
\end{aligned}$$

Next we list some parametrizations R_1, \dots, R_{15} in $\mathbb{Q}(t) \setminus \mathbb{Z}[t]$ (some of them have in their image some Büchi sequences of integers, but always finitely many). Note that the four components of each

parametrization always come with the same denominator (as reduced fractions). Note also that each of these parametrizations generate infinitely many new parametrizations by iterating the map ζ .

$$\begin{aligned}
(4t^2 + 4)R_1(t) &= \begin{cases} t^9 + 4t^8 - t^7 - 10t^6 + 9t^5 - 23t^3 + 10t^2 - 10t - 4 \\ t^9 + 4t^8 + t^7 - 4t^6 + 13t^5 + 8t^4 - 9t^3 + 16t^2 + 2t \\ t^9 + 4t^8 + 3t^7 + 2t^6 + 13t^5 + 8t^4 - 3t^3 + 6t^2 + 10t - 4 \\ t^9 + 4t^8 + 5t^7 + 8t^6 + 9t^5 + 19t^3 + 4t^2 - 10t + 8 \end{cases} \\
(4t^2 + 4)R_2(t) &= \begin{cases} t^9 - 4t^8 - t^7 + 10t^6 + 9t^5 - 23t^3 - 10t^2 - 10t + 4 \\ t^9 - 4t^8 + t^7 + 4t^6 + 13t^5 - 8t^4 - 9t^3 - 16t^2 + 2t \\ t^9 - 4t^8 + 3t^7 - 2t^6 + 13t^5 - 8t^4 - 3t^3 - 6t^2 + 10t + 4 \\ t^9 - 4t^8 + 5t^7 - 8t^6 + 9t^5 + 19t^3 - 4t^2 - 10t - 8 \end{cases} \\
8tR_3(t) &= \begin{cases} t^6 - 10t^5 + 38t^4 - 84t^3 + 120t^2 - 96t + 48 \\ t^6 - 8t^5 + 30t^4 - 68t^3 + 96t^2 - 88t + 48 \\ t^6 - 6t^5 + 18t^4 - 44t^3 + 72t^2 - 80t + 48 \\ t^6 - 4t^5 + 2t^4 + 12t^3 - 48t^2 + 72t - 48 \end{cases} \\
4tR_4(t) &= \begin{cases} t^6 - 17t^5 + 113t^4 - 369t^3 + 600t^2 - 420t + 96 \\ t^6 - 15t^5 + 93t^4 - 295t^3 + 480t^2 - 352t + 96 \\ t^6 - 13t^5 + 69t^4 - 193t^3 + 312t^2 - 284t + 96 \\ t^6 - 11t^5 + 41t^4 - 39t^3 - 96t^2 + 216t - 96 \end{cases} \\
(4t^2 - 28)R_5(t) &= \begin{cases} t^7 - 6t^6 - 3t^5 + 60t^4 - 33t^3 - 162t^2 + 107t + 60 \\ t^7 - 4t^6 - 5t^5 + 36t^4 - 17t^3 - 88t^2 + 93t + 32 \\ t^7 - 2t^6 - 11t^5 + 12t^4 + 55t^3 - 14t^2 - 117t + 4 \\ t^7 - 21t^5 + 12t^4 + 111t^3 - 60t^2 - 163t + 24 \end{cases} \\
(4t^2 - 4t)R_6(t) &= \begin{cases} t^7 - 16t^6 + 98t^5 - 300t^4 + 506t^3 - 480t^2 + 240t - 48 \\ t^7 - 14t^6 + 80t^5 - 240t^4 + 406t^3 - 396t^2 + 212t - 48 \\ t^7 - 12t^6 + 58t^5 - 156t^4 + 270t^3 - 296t^2 + 184t - 48 \\ t^7 - 10t^6 + 32t^5 - 24t^4 - 70t^3 + 180t^2 - 156t + 48 \end{cases} \\
(8t^2 - 16t)R_7(t) &= \begin{cases} t^7 - 14t^6 + 86t^5 - 300t^4 + 644t^3 - 840t^2 + 624t - 192 \\ t^7 - 12t^6 + 70t^5 - 240t^4 + 516t^3 - 688t^2 + 544t - 192 \\ t^7 - 10t^6 + 50t^5 - 156t^4 + 340t^3 - 504t^2 + 464t - 192 \\ t^7 - 8t^6 + 26t^5 - 24t^4 - 76t^3 + 288t^2 - 384t + 192 \end{cases} \\
(8t^3 + 48t^2 + 32t)R_8(t) &= \begin{cases} t^8 + 12t^7 + 50t^6 + 96t^5 + 144t^4 + 240t^3 + 304t^2 + 288t + 128 \\ t^8 + 14t^7 + 78t^6 + 240t^5 + 488t^4 + 664t^3 + 608t^2 + 384t + 128 \\ t^8 + 16t^7 + 102t^6 + 336t^5 + 656t^4 + 896t^3 + 848t^2 + 480t + 128 \\ t^8 + 18t^7 + 122t^6 + 408t^5 + 792t^4 + 1080t^3 + 1024t^2 + 576t + 128 \end{cases} \\
(12t^2 + 12)R_9(t) &= \begin{cases} t^7 - 6t^6 + 21t^5 - 60t^4 + 111t^3 - 162t^2 + 163t - 108 \\ t^7 - 4t^6 + 19t^5 - 44t^4 + 95t^3 - 136t^2 + 149t - 96 \\ t^7 - 2t^6 + 13t^5 - 28t^4 + 71t^3 - 110t^2 + 131t - 84 \\ t^7 + 3t^5 + 12t^4 - 33t^3 + 84t^2 - 107t + 72 \end{cases}
\end{aligned}$$

$$(3t^2 - 15)R_{10}(t) = \begin{cases} t^7 - 6t^6 + 3t^5 + 30t^4 - 33t^3 - 27t^2 + 37t - 27 \\ t^7 - 4t^6 + t^5 + 16t^4 - 25t^3 - 10t^2 + 47t - 12 \\ t^7 - 2t^6 - 5t^5 + 2t^4 + 23t^3 + 7t^2 - 43t + 3 \\ t^7 - 15t^5 + 12t^4 + 39t^3 - 24t^2 - 17t - 18 \end{cases}$$

$$(t^3 - t^2 - t + 1)R_{11}(t) = \begin{cases} t^4 - 6t^2 - 28t - 39 \\ t^4 + 3t^3 + 11t^2 + 25t + 32 \\ t^4 + 6t^3 + 20t^2 + 22t + 23 \\ t^4 + 9t^3 + 21t^2 + 35t + 6 \end{cases}$$

$$4t^2R_{12}(t) = \begin{cases} t^3 + 3t^2 + 9t - 9 \\ t^3 + t^2 - 3t + 9 \\ t^3 - t^2 - 3t - 9 \\ t^3 - 3t^2 + 9t + 9 \end{cases}$$

$$4t^2R_{13}(t) = \begin{cases} t^3 - 12t^2 - 72t + 288 \\ t^3 - 4t^2 - 24t + 288 \\ t^3 + 4t^2 - 24t - 288 \\ t^3 + 12t^2 - 72t - 288 \end{cases}$$

$$(240t^3 - 588240t^2 + 462160080t - 113856219120)R_{14}(t) =$$

$$\begin{cases} t^4 - 1508t^3 - 616026t^2 + 1404632668t - 262572118559 \\ t^4 - 2228t^3 + 1379094t^2 - 321457172t + 190871636401 \\ t^4 - 2948t^3 + 2913414t^2 - 815367812t - 172760883839 \\ t^4 - 3668t^3 + 3986934t^2 - 1072427252t - 221781743279 \end{cases}$$

$$(240t^3 + 45360t^2 - 15574320t - 3067284720)R_{15}(t) =$$

$$\begin{cases} t^4 - 1508t^3 - 616026t^2 + 381656668t + 123089833441 \\ t^4 - 788t^3 - 249546t^2 + 327100108t + 100683524881 \\ t^4 - 68t^3 - 343866t^2 - 264749252t - 71708114879 \\ t^4 + 652t^3 - 898986t^2 - 398563412t - 12874461839. \end{cases}$$

5 Some basic properties of the sequence $(\xi(n, t))_n$

In this section we prove that we need only studying $\xi(n, t)$ for $t \geq 0$ and we show that for fix $t \geq 0$, the sequences $(\xi_i(n, t))_i$ and $(\xi_i(n, t))_n$ are strictly increasing sequences of positive integers.

A straightforward computation shows that for all $n \geq 0$ and $t \in \mathbb{Z}$ we have

$$\xi_4(n, t) = -\xi_1(n, -t - 5) \quad \text{and} \quad \xi_3(n, t) = -\xi_2(n, -t - 5). \quad (5)$$

In particular, since

$$\xi(1, -2) = (0, 1, -2, -3) \quad \text{and} \quad \xi(1, -1) = (-3, 4, 5, 6)$$

are trivial sequences, also $\xi(1, -3)$ and $\xi(1, -4)$ are trivial sequences.

Lemma 5.1. *For each $n \geq 0$, the Büchi sequences $\xi(n, -4)$, $\xi(n, -3)$, $\xi(n, -2)$ and $\xi(n, -1)$ are trivial sequences.*

Proof. From Theorem 3.2, we have for $t = -1$

$$\xi_i(n+2, -1) = 2\xi_i(n+1, -1) - \xi_i(n, -1)$$

for each $i = 1, 2, 3, 4$, with initial values for $n = 0, 1$ (reminding that $\xi(0, t) = (t+1, t+2, t+3, t+4)$):

$$\begin{aligned} \xi_1(0, -1) &= 0 & \xi_1(1, -1) &= -3 \\ \xi_2(0, -1) &= 1 & \xi_2(1, -1) &= 4 \\ \xi_3(0, -1) &= 2 & \xi_3(1, -1) &= 5 \\ \xi_4(0, -1) &= 3 & \xi_4(1, -1) &= 6; \end{aligned}$$

and for $t = -2$:

$$\xi_i(n+2, -2) = -2\xi_i(n+1, -2) - \xi_i(n, -2)$$

for each $i = 1, 2, 3, 4$, with initial values for $n = 0, 1$

$$\begin{aligned} \xi_1(0, -2) &= -1 & \xi_1(1, -2) &= 0 \\ \xi_2(0, -2) &= 0 & \xi_2(1, -2) &= 1 \\ \xi_3(0, -2) &= 1 & \xi_3(1, -2) &= -2 \\ \xi_4(0, -2) &= 2 & \xi_4(1, -2) &= -3. \end{aligned}$$

Solving the eight recurrence relations above, we obtain :

$$\xi(n, -1) = (-3n, 3n+1, 3n+2, 3n+3) \quad \text{and} \quad \xi(n, -2) = (-1)^n(n-1, -n, n+1, n+2)$$

which are clearly trivial sequences. From Equations (5), we have

$$\begin{aligned} \xi_1(n, -3) &= -\xi_4(n, -2) & \xi_2(n, -3) &= -\xi_3(n, -2) \\ \xi_4(n, -3) &= -\xi_1(n, -2) & \xi_3(n, -3) &= -\xi_2(n, -2), \end{aligned}$$

hence

$$\xi(n, -3) = (-1)^n(-n-2, -n-1, n, -n+1),$$

and

$$\begin{aligned} \xi_1(n, -4) &= -\xi_4(n, -1) & \xi_2(n, -4) &= -\xi_3(n, -1) \\ \xi_4(n, -4) &= -\xi_1(n, -1) & \xi_3(n, -4) &= -\xi_2(n, -1), \end{aligned}$$

hence

$$\xi(n, -4) = (-3n-3, -3n-2, -3n-1, 3n)$$

which are also clearly trivial sequences. □

Remark 5.2. *We deduce from Equations (5) and Lemma 5.1 that it is enough to study the parametrizations $\xi(n, t)$ for $t \geq 0$.*

Lemma 5.3. *If (u_n) is a sequence of integers satisfying $u_{n+2} = \alpha u_{n+1} - u_n$ for each $n \geq 0$, with $\alpha \geq 2$, and $u_1 > u_0 > 0$, then for all $n \geq 1$ it satisfies $u_{n+1} > (\alpha - 1)u_n > 0$.*

Proof. We have

$$u_2 = \alpha u_1 - u_0 = (\alpha - 1)u_1 + u_1 - u_0 > (\alpha - 1)u_1 > 0.$$

Suppose that $u_{n+1} > (\alpha - 1)u_n > 0$ for some $n \geq 1$. We have

$$u_{n+2} = \alpha u_{n+1} - u_n > \alpha u_{n+1} - \frac{u_{n+1}}{\alpha - 1} \geq (\alpha - 1)u_{n+1}.$$

□

Corollary 5.4. *For each $t \geq 0$ and for each $i = 1, \dots, 4$, we have*

$$\xi_i(n+1, t) > (2t^2 + 10t + 9)\xi_i(n, t).$$

Proof. Fix $t \geq 0$. We apply Lemma 5.3 to the sequence $u_n = \xi_i(n, t)$ for each $i = 1, \dots, 4$. By Item 3 of Theorem 3.2, u_n satisfy the recurrence relation $u_{n+2} = \alpha u_{n+1} - u_n$, with

$$\alpha = f(t) = 2t^2 + 10t + 10 \geq 2.$$

By Items 1 and 2 of Theorem 3.2, we have

$$u_1 = \begin{cases} \xi_1(1, t) = 2t^3 + 12t^2 + 19t + 6 > t + 1 = \xi_1(0, t) = u_0 > 0 & \text{if } i = 1 \\ \xi_2(1, t) = 2t^3 + 14t^2 + 31t + 23 > t + 2 = \xi_2(0, t) = u_0 > 0 & \text{if } i = 2 \\ \xi_3(1, t) = 2t^3 + 16t^2 + 41t + 32 > t + 3 = \xi_3(0, t) = u_0 > 0 & \text{if } i = 3 \\ \xi_4(1, t) = 2t^3 + 18t^2 + 49t + 39 > t + 4 = \xi_4(0, t) = u_0 > 0 & \text{if } i = 4 \end{cases}$$

and we conclude in each case by Lemma 5.3. \square

Lemma 5.5. *If (v_n) and (w_n) are sequences of integers both satisfying the same recurrence relation $u_{n+2} = \alpha u_{n+1} - u_n$ for each $n \geq 0$, with $\alpha \geq 2$, and $u_1 > u_0 > 0$, and such that $w_0 \geq v_0$ and $w_1 - w_0 > v_1 - v_0$, then for all $n \geq 1$ we have $w_{n+1} - v_{n+1} > (\alpha - 1)(w_n - v_n) > 0$.*

Proof. We have

$$\begin{aligned} w_2 - v_2 &= \alpha w_1 - w_0 - (\alpha v_1 - v_0) \\ &= (\alpha - 1)(w_1 - v_1) + w_1 - w_0 - (v_1 - v_0) \\ &> (\alpha - 1)(w_1 - v_1) \\ &> (\alpha - 1)(w_0 - v_0) \geq 0. \end{aligned}$$

If for some $n \geq 1$ we have $w_{n+1} - v_{n+1} > (\alpha - 1)(w_n - v_n) > 0$ then

$$\begin{aligned} w_{n+2} - v_{n+2} &= \alpha w_{n+1} - w_n - (\alpha v_{n+1} - v_n) \\ &= \alpha(w_{n+1} - v_{n+1}) - (w_n - v_n) \\ &> \alpha(w_{n+1} - v_{n+1}) - \frac{w_{n+1} - v_{n+1}}{\alpha - 1} \\ &> (\alpha - 1)(w_{n+1} - v_{n+1}) > 0. \end{aligned}$$

\square

Corollary 5.6. *For each n and each $t \geq 0$, the sequence $\xi(n, t)$ is a strictly increasing non-trivial Büchi sequence of positive integers. Moreover, for each $i = 1, 2, 3$ and for each $n \geq 1$ we have*

$$\xi_{i+1}(n+1, t) - \xi_i(n+1, t) > (2t^2 + 10t + 9)(\xi_{i+1}(n, t) - \xi_i(n, t)).$$

Proof. Fix $t \geq 0$. We will apply Lemma 5.5 to the sequences $v_n = \xi_1(n, t)$ and $w_n = \xi_2(n, t)$. By Item 3 of Theorem 3.2, both v_n and w_n satisfy the recurrence relation $u_{n+2} = \alpha u_{n+1} - u_n$, with

$$\alpha = f(t) = 2t^2 + 10t + 10 \geq 2.$$

By Items 1 and 2 of Theorem 3.2, we have

$$\begin{aligned} v_1 &= \xi_1(1, t) = 2t^3 + 12t^2 + 19t + 6 > t + 1 = \xi_1(0, t) = v_0 > 0, \\ w_1 &= \xi_2(1, t) = 2t^3 + 14t^2 + 31t + 23 > t + 2 = \xi_2(0, t) = w_0 > 0, \end{aligned}$$

$$w_0 = \xi_2(0, t) = t + 2 > t + 1 = \xi_1(0, t) = v_0,$$

and

$$w_1 - w_0 = 2t^3 + 14t^2 + 30t + 21 > 2t^3 + 12t^2 + 18t + 5 = v_1 - v_0$$

so all the hypothesis of Lemma 5.5 are satisfied and we deduce that $\xi_2(n, t) - \xi_1(n, t)$ is a positive integer for each $n \geq 0$ and that for each $n \geq 1$ we have

$$\xi_2(n+1, t) - \xi_1(n+1, t) > (f(t) - 1)(\xi_2(n, t) - \xi_1(n, t)).$$

The two other cases are verified similarly. \square

6 A family of curves associated to length 5 solutions

Each length 4 integer sequence (x_1, x_2, x_3, x_4) might extend to the right or to the left. For given integers $n \geq 1$ and $t \geq 0$, a Büchi sequence $\xi(n, t)$ extends to the right if and only if $2\xi_4^2(n, t) - \xi_3^2(n, t) + 2$ is a square, and it extends to the left if and only if $2\xi_1^2(n, t) - \xi_2^2(n, t) + 2$ is a square. So for each $n \geq 1$, we want to know whether or not the curves

$$y^2 = 2\xi_4^2(n, t) - \xi_3^2(n, t) + 2 \quad (C_n^r)$$

and

$$y^2 = 2\xi_1^2(n, t) - \xi_2^2(n, t) + 2 \quad (C_n^\ell)$$

have integer points at all. The polynomials on the right have degree $2(2n+1) = 4n+2$. Unfortunately, we cannot prove that all these curves are hyperelliptic (we verified[†] it only up to $n = 18$). If it were the case then we would already know that each of them has only finitely many integer points. If there are only finitely many Büchi sequences of length 5, as we suspect, then all but finitely many of these curves has no integer point at all.

Here is the equation for (C_1^r)

$$y^2 = 4t^6 + 80t^5 + 620t^4 + 2400t^3 + 4905t^2 + 5020t + 2020,$$

the one for (C_2^r)

$$16t^{10} + 480t^9 + 6240t^8 + 46400t^7 + 218812t^6 + 684120t^5 + 1436320t^4 + 1999600t^3 + 1766797t^2 + 894990t + 197505,$$

and the one for (C_3^r)

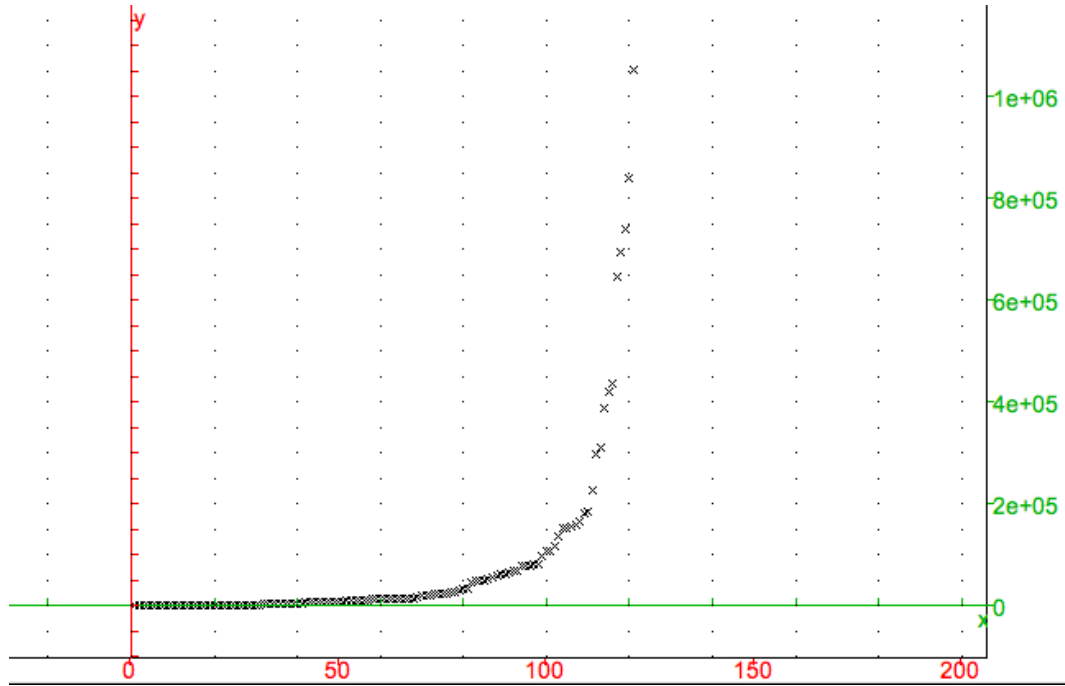
$$64t^{14} + 2560t^{13} + 46400t^{12} + 505600t^{11} + 3702416t^{10} + 19280000t^9 + 73635280t^8 + 209537600t^7 + 446403560t^6 + 708503520t^5 + 824619920t^4 + 682516400t^3 + 379789209t^2 + 127204040t + 19353040.$$

7 A List of non-parametrized integer points on X_4

In this section we list the strictly increasing sequences that are not given by any of the parametrizations presented in this paper. The first column is just the number of the line of the matrix. The graph is a plot of the two first columns. The quantity of points that we are not able to parametrize seems to get exponentially towards zero.

1	59	630	889	1088	48	7414	16875	22684	27283
2	83	516	725	886	49	7594	10997	13572	15731
3	108	6643	9394	11505	50	7871	12162	15293	17884
4	108	707	994	1215	51	8562	17089	22600	27009
5	177	878	1229	1500	52	9343	26408	36159	43790
6	240	839	1162	1413	53	9741	19460	25739	30762
7	287	11838	16739	20500	54	9752	25249	34350	41501
8	311	752	1017	1226	55	10888	25561	34470	41509
9	334	3693	5212	6379	56	11358	47107	65644	79995
10	386	6237	8812	10789	57	12129	18232	22753	26514
11	419	11020	15579	19078	58	12539	21430	27591	32608
12	430	801	1048	1247	59	12710	46491	64508	78493
13	477	3572	5029	6150	60	13305	44986	62213	75612
14	510	1699	2348	2853	61	13500	29971	40178	48273
15	514	1537	2112	2561	62	13811	38380	52491	63542
16	570	7879	11128	13623	63	13835	33596	45453	54802
17	601	4832	6807	8326	64	13836	25693	33598	39969
18	862	1713	2264	2705	65	14416	40737	55778	67549
19	883	25566	36145	44264	66	14843	26758	34809	41320
20	916	26605	37614	46063	67	15369	52022	71947	87444
21	1346	20353	28752	35201	68	15451	47988	66083	80194
22	1546	5257	7272	8839	69	18793	33744	43865	52054
23	1574	2693	3468	4099	70	20476	44445	59426	71327
24	1616	3353	4458	5339	71	21648	38497	49954	59235
25	1674	2695	3424	4023	72	21924	32243	39982	46449
26	1766	8837	12372	15101	73	22377	45328	60071	71850
27	1812	11587	16286	19905	74	23173	49926	66695	80024
28	2066	6963	9628	11701	75	23174	56283	76148	91811
29	2437	13062	18311	22360	76	25079	34122	41227	47276
30	2477	15876	22315	27274	77	27283	57918	77231	92600
31	2636	20685	29134	35633	78	27699	38828	47413	54666
32	3048	5047	6454	7605	79	31659	51412	65453	76974
33	3051	11578	16087	19584	80	33426	58483	75652	89589
34	3247	9746	13395	16244	81	34030	59119	76368	90383
35	3333	36682	51769	63360	82	45007	85256	111855	133246
36	3673	5478	6821	7940	83	49040	61729	72222	81373
37	4090	5701	6948	8003	84	50430	70781	86468	99717
38	4743	36806	51835	63396	85	51077	89226	115385	136624
39	5148	12253	16546	19935	86	53119	70562	84477	96404
40	5331	15988	21973	26646	87	55506	72097	85528	97119
41	5781	22342	31063	37824	88	58599	87328	108713	126534
42	6449	25358	35277	42964	89	62429	86532	105253	121114
43	6504	18065	24706	29907	90	63626	118165	154524	183827
44	6756	33773	47282	57711	91	64776	98815	123826	144573
45	7104	9823	11938	13731	92	68986	106617	134072	156791
46	7234	24447	33808	41089	93	70143	94792	114241	130830
47	7386	17033	22928	27591	94	77391	92440	105361	116862

95	78741	128278	163433	192264
96	79292	91693	102606	112465
97	80251	100090	116601	131048
98	81770	131541	167092	196307
99	98804	118755	135806	150943
100	107366	169275	213964	250813
101	108523	139124	164115	185774
102	117178	144071	166680	186569
103	138004	167365	192294	214343
104	154097	200846	238605	271156
105	154097	200846	238605	271156
106	155730	226399	279752	324447
107	158435	195324	226277	253478
108	165267	222418	267631	306240
109	183122	235379	277980	314869
110	186101	246132	294157	335374
111	225341	270018	308287	342304
112	297422	352179	399500	441781
113	311680	401551	474702	537997
114	388048	447801	500470	548101
115	421884	499235	566114	625887
116	435682	484931	529620	570821
117	646914	739327	821408	896001
118	695001	761728	823063	880134
119	740566	869223	981152	1081559
120	839833	974682	1093019	1199740
121	1052749	1157218	1253007	1341976



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